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TABLES OF CHARACTERISTIC FUNCTIONS REPRESENTING NORMAL MODES OF VIBRATION OF A BEAM

By

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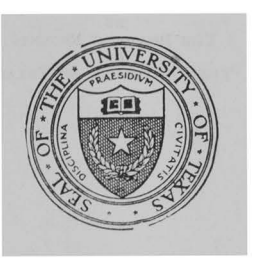

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The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

Sam Houston

Cultivated mind is the guardian genius of Democracy, and while guided and controlled by virtue, the noblest attribute of man. It is the only dictator that freemen acknowledge, and the only security which freemen desire.

Mirabeau B. Lamar

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CONTENTS

INTRODUCTION AND DEFINITIONS

A beam can vibrate laterally at an infinite number of natural frequencies. For each frequency there is a definite shape in which the beam will deflect while vibrating harmonically; this shape is called a *normal mode* of vibration of the beam. The mathematical expressions which define the normal modes are called *characteristic functions*. For each type of beam with specified end conditions there is an infinite set of these functions.

It is the purpose of this bulletin to provide tables of the characteristic functions for nearly all common types of beams. Such tables are needed in obtaining complete numerical solutions for many problems involving beam vibrations. Furthermore, characteristic functions may be used advantageously in the solution of a number of other problems in the theory of structures.

The different types of beams will be identified by a compound adjective which describes the end conditions. Thus a "clamped-free" beam is one which is rigidly clamped or fixed at the end $x = o$, and entirely free or unsupported at the end $x = l$; a "supported-clamped" beam is one which is simply supported at the end $x = o$, and clamped at the end $x = l$; and similarly for other types. There are six common types of beams, namely, (1) clamped-clamped, (2) clamped-free, (3) clamped-supported, (4) free-free, (5) free-supported, and (6) supported-supported.

REFERENCES TO THEORY AND APPLICATIONS

The method of determining the natural frequencies and modes of vibration of a beam and a detailed derivation of the corresponding characteristic functions are given in standard texts such as Refs. 1, 2, and 3 in the bibliography. All the functions tabulated herein are for a *uniform* beam whose vibrations are governed by the well known differential equation,*

$$EI \frac{\partial^4 y}{\partial x^4} + \rho \frac{\partial^2 y}{\partial t^2} = 0$$

The mathematical properties of the characteristic functions are discussed in the references cited above where it is shown that each of the functions for a given type of beam satisfies the differential equation

$$\frac{d^4 \phi_n}{dx^4} = \beta_n^4 \phi_n , \quad \text{or} \quad \phi_n'''' = \phi_n$$

and satisfies the boundary conditions corresponding to the end conditions of the given type of beam. It is also shown that the set of characteristic

*The notation used in this bulletin is defined in the section on nomenclature on page 10.

functions ϕ_n, $(n = 1, 2, 3, \ldots)$, for any one of the six common types of beams satisfy the so-called *orthogonality relations*, namely,

$$\int_0^l \phi_n \phi_m \, dx = 0 \qquad (m \neq n)$$

$$\int_0^l \phi_n^2 \, dx = \text{constant}$$

and consequently the set of functions is orthogonal in the interval

$$o \leq x \leq l.$$

In the solution of differential equations by means of characteristic functions it is often necessary to expand an arbitrary function, generally the load function, in a linear series of the characteristic functions. The method of doing this is explained in Ref. 3 and 4. Such series are actually a generalized form of the familiar Fourier sine and cosine series. A discussion of the validity of this generalized series representation is given in Ref. 5.

Characteristic functions are used to solve a number of technically important problems such as the forced vibration of a beam both with and without damping, vibration of a beam on an elastic foundation, motion of a beam under the action of a moving force, the bending of a beam on an elastic foundation, and the bending of the wall of a circular tank. A general discussion of the use of these functions in such problems has been given by Karman (Ref. 4). The application to forced vibrations of a beam is shown in Ref. 2 and 3. Inglis (Ref. 6) discusses a great number of different applications in the particular case of a simply supported beam for which solutions may be obtained in the form of sine series. His methods of analysis can be extended to beams with other end conditions by the use of the characteristic functions having the appropriate boundary values. The transient motion of a beam following a sudden impact is solvable in terms of characteristic functions as shown in Ref. 7. The application of characteristic functions in solving for the frequencies of a composite system composed of a beam carrying a spring supported mass is discussed in Ref. 8.

The characteristic functions can also be used in the Rayleigh-Ritz energy method for solving such problems as the bending, buckling, and vibration of rectangular plates. The application to the buckling of clamped rectangular plates has been carried out by Maulbetsch (Ref. 9). W. Ritz (Ref. 10) used the characteristic functions for a free-free beam to investigate the vibrations of a square plate with free edges.

DESCRIPTIONS OF TABLES

The tables herein give the values of the characteristic function and its first three derivatives for each of the first five modes ($n = 1, 2, 3, 4, 5$), of three different types of beams, namely,

Table 1 Clamped-clamped beam
Table 2 Clamped-free beam
Table 3 Clamped-supported beam

The functions are tabulated to five decimal places at intervals of the argument corresponding to 1/50 of the beam length, that is, 0.02 l.

It may be shown that the characteristic function for a *free-free* beam is the same as the second derivative of the characteristic function for a clamped-clamped beam, and that the characteristic function for a *free-supported* beam is the same as the second derivative of the characteristic function for a clamped-supported beam. Therefore, the values of the functions for these two additional cases can be obtained directly from the tables. The only common type of beam not included is the supported-supported beam for which the characteristic function is the ordinary trigonometric sine.

Preceding each table is a brief summary of data which includes the mathematical expressions for the characteristic function and its derivatives, the numerical values of the parameters α_n and β_n, and various powers thereof which are useful in numerical calculations.

It will be noted that the function and its first three derivatives are different. However, higher derivatives repeat the cycle, that is,

$$\phi_n^{IIII} = \phi_n, \quad \phi_n^{V} = \phi_n^{I}, \quad \phi_n^{VI} = \phi_n^{II}, \quad \text{etc.}$$

It should also be noted that the essential properties of the characteristic functions are not altered if each one is multiplied by a constant. The only effect would be to multiply the tabular values by that constant. In general it will be found that the expressions for the characteristic functions given in different references differ from one another by some constant factor. The particular forms of the expressions for the characteristic functions used in this bulletin have been chosen so that for all the different types of beams,

$$\int_0^l \phi_n^2 \, dx = l$$

BIBLIOGRAPHY

1. Rayleigh, Lord, *Theory of Sound*, first American edition, Dover Publications, New York, 1945, pp. 225–305.

2. Timoshenko, S., *Vibration Problems in Engineering,* second edition, D. Van Nostrand Co., New York, 1937, pp. 331–376.

3. Karman, T., and Biot, M. A., *Mathematical Methods in Engineering,* McGraw-Hill Book Co., New York, 1940, pp. 286–295 and 347–352.

4. Karman, T., "Use of Orthogonal Functions in Structural Problems," *Stephen Timoshenko 60th Anniversary Volume,* Macmillan Co., New York, 1938.

5. Churchill, R. V., *Fourier Series and Boundary Value Problems,* McGraw-Hill Book Co., New York, 1941.

6. Inglis, C. E., *A Mathematical Treatise on Vibrations in Railway Bridges,* Cambridge University Press, Cambridge, 1934.

7. Morse, P. M., *Vibration and Sound,* McGraw-Hill Book Co., New York, 1936, pp. 111–131.

8. Young, D., "Vibration of a Beam with Concentrated Mass, Spring, and Dashpot," *Journal of Applied Mechanics,* Trans. A.S.M.E., vol. 70, March, 1948, pp. 65–72.

9. Maulbetsch, J. L., "Buckling of Compressed Rectangular Plates with Built-in Edges," *Journal of Applied Mechanics,* Trans. A.S.M.E. vol. 59, 1937, pp. A59–A62.

10. Ritz, W., *Gesammelte Werke Walther Ritz,* Gauthier-Villars, Paris, 1911. A brief summary of Ritz's solution for the vibration of a rectangular plate is given in Reference 2, p. 425.

NOMENCLATURE

The nomenclature used is summarized below. Any consistent set of units may be used for the physical quantities, however, for convenience, the units of such quantities are noted in terms of the inch-pound-second system which is commonly employed in engineering.

x = co-ordinate distance along the longitudinal axis of beam, in.

y = lateral deflection of beam axis, in.

l = length of beam, in.

t = time, sec.

E = modulus of elasticity of beam material, lb./in.^{2}

I = moment of inertia of a cross-section of a beam about its neutral axis, in.^{4}

n = 1,2,3,4, . . . = any interger, generally used as a subscript to indicate the order of the mode of vibration.

ρ = mass per unit length of a beam, lb. sec.^{2}/in.^{2}

$\omega_n = \beta_n^2 \sqrt{EI/\rho}$ = natural circular frequency of the n-th mode of vibration of a beam, radians per sec.

β_n =the characteristic number; it is related to the frequency ω_n by the relation immediately above. Numerical values for each type of beam are given in the summary of data preceding each table.

α_n =a parameter occurring in the expressions for the characteristic functions. Numerical values for each type of beam are given in the summary of data preceding each table.

$\phi_n(x)$ =characteristic function which is defined by the analytical expression given in the summary of data preceding each table. Derivatives with respect to $\beta_n x$ are denoted by prime marks, thus

$$\phi'_n = \frac{1}{\beta_n}\frac{d\phi_n}{dx}, \quad \phi''_n = \frac{1}{\beta_n^2}\frac{d^2\phi_n}{dx^2}, \quad \phi'''_n = \frac{1}{\beta_n^3}\frac{d^3\phi_n}{dx^3}$$

Data for Clamped-Clamped Beam and Free-Free Beam

A. CLAMPED-CLAMPED BEAM

Characteristic function and its derivatives

$$\phi_n = \text{Cosh}\,\beta_n x - \cos\beta_n x - \alpha_n(\text{Sinh}\,\beta_n x - \sin\beta_n x)$$

$$\frac{1}{\beta_n}\frac{d\phi_n}{dx} = \phi_n' = \text{Sinh}\,\beta_n x + \sin\beta_n x - \alpha_n(\text{Cosh}\,\beta_n x - \cos\beta_n x)$$

$$\frac{1}{\beta_n^2}\frac{d^2\phi_n}{dx^2} = \phi_n'' = \text{Cosh}\,\beta_n x + \cos\beta_n x - \alpha_n(\text{Sinh}\,\beta_n x + \sin\beta_n x)$$

$$\frac{1}{\beta_n^3}\frac{d^3\phi_n}{dx^3} = \phi_n''' = \text{Sinh}\,\beta_n x - \sin\beta_n x - \alpha_n(\text{Cosh}\,\beta_n x + \cos\beta_n x)$$

Boundary values

$$\phi_n(0) = \phi_n'(0) = 0$$

$$\phi_n(\ell) = \phi_n'(\ell) = 0$$

Values of α_n and $\beta_n\ell$ and various powers

n	$\beta_n\ell$	$(\beta_n\ell)^2$	$(\beta_n\ell)^3$	$(\beta_n\ell)^4$
1	4.7300 408	22.3732 86	105.8265 6	500.5639
2	7.8532 046	61.6728 22	484.3293 9	3 803.5370
3	10.9956 078	120.9033 91	1329.4062 7	14 617.6299
4	14.1371 655	199.8594 48	2825.4461 0	39 943.7991
5	17.2787 596	298.5555 33	5158.6692 9	89 135.4065

n	α_n	$\frac{\omega_n}{\omega_1} = \frac{\beta_n^2}{\beta_1^2}$	$\frac{\omega_n^2}{\omega_1^2} = \frac{\beta_n^4}{\beta_1^4}$
1	0.9825 0221 58	1.0	1.0
2	1.0007 7731 1	2.7565 384	7.5985 041
3	0.9999 6645 01	5.4039 175	29.2023 239
4	1.0000 0145 0	8.9329 501	79.7975 983
5	0.9999 9993 73	13.3442 863	178.0699 762

For $n>5$

$\beta_n\ell \doteq (2n-1)\pi/2$

$\alpha_n \doteq 1.0$

B. FREE-FREE BEAM

Characteristic function

$$\phi_n = \text{Cosh}\beta_n x + \cos\beta_n x - \alpha_n (\text{Sinh}\beta_n x + \sin\beta_n x)$$

where α_n and β_n are the same as for a clamped-clamped beam.

The characteristic function for a free-free beam is the same as the second derivative of a clamped-clamped beam, that is

$$\begin{aligned}
\phi_n \text{ (free-free)} &= \phi_n'' \text{ (clamped-clamped)} \\
\phi_n' \text{ (free-free)} &= \phi_n''' \text{ (clamped-clamped)} \\
\phi_n'' \text{ (free-free)} &= \phi_n \text{ (clamped-clamped)} \\
\phi_n''' \text{ (free-free)} &= \phi_n' \text{ (clamped-clamped)}
\end{aligned}$$

Boundary values

$$\begin{aligned}
\phi_n''(0) &= \phi_n'''(0) = 0 \\
\phi_n''(\ell) &= \phi_n'''(\ell) = 0
\end{aligned}$$

TABLE 1

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-CLAMPED BEAM
First Mode

$\frac{x}{\ell}$	ϕ_1	$\phi_1' = \frac{1}{\beta_1}\frac{d\phi_1}{dx}$	$\phi_1'' = \frac{1}{\beta_1^2}\frac{d^2\phi_1}{dx^2}$	$\phi_1''' = \frac{1}{\beta_1^3}\frac{d^3\phi_1}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 1.96500
0.02	0.00867	0.18041	1.81412	- 1.96473
0.04	0.03358	0.34324	1.62832	- 1.96285
0.06	0.07306	0.48850	1.44284	- 1.95792
0.08	0.12545	0.61624	1.25802	- 1.94862
0.10	0.18910	0.72655	1.07433	- 1.93383
0.12	0.26237	0.81956	0.89234	- 1.91254
0.14	0.34363	0.89546	0.71270	- 1.88393
0.16	0.43126	0.95451	0.53615	- 1.84732
0.18	0.52370	0.99702	0.36346	- 1.80219
0.20	0.61939	1.02342	0.19545	- 1.74814
0.22	0.71684	1.03418	0.03300	- 1.68494
0.24	0.81459	1.02986	- 0.12305	- 1.61250
0.26	0.91124	1.01113	- 0.27180	- 1.53085
0.28	1.00546	0.97870	- 0.41240	- 1.44017
0.30	1.09600	0.93338	- 0.54401	- 1.34074
0.32	1.18168	0.87608	- 0.66581	- 1.23296
0.34	1.26141	0.80774	- 0.77704	- 1.11735
0.36	1.33419	0.72992	- 0.87699	- 0.99452
0.38	1.39913	0.64219	- 0.96500	- 0.86516
0.40	1.45545	0.54723	- 1.04050	- 0.73007
0.42	1.50247	0.44574	- 1.10297	- 0.59008
0.44	1.53962	0.33897	- 1.15202	- 0.44611
0.46	1.56647	0.22821	- 1.18728	- 0.29911
0.48	1.58271	0.11478	- 1.20854	- 0.15007
0.50	1.58815	0.00000	- 1.21565	0.00000
0.52	1.58271	- 0.11478	- 1.20854	0.15007
0.54	1.56647	- 0.22821	- 1.18728	0.29911
0.56	1.53962	- 0.33897	- 1.15202	0.44611
0.58	1.50247	- 0.44574	- 1.10297	0.59008
0.60	1.45545	- 0.54723	- 1.04050	0.73007
0.62	1.39913	- 0.64219	- 0.96500	0.86516
0.64	1.33419	- 0.72992	- 0.87699	0.99452
0.66	1.26141	- 0.80774	- 0.77704	1.11735
0.68	1.18168	- 0.87608	- 0.66581	1.23296
0.70	1.09600	- 0.93338	- 0.54401	1.34074
0.72	1.00546	- 0.97870	- 0.41240	1.44017
0.74	0.91124	- 1.01113	- 0.27180	1.53085
0.76	0.81459	- 1.02986	- 0.12305	1.61250
0.78	0.71684	- 1.03418	0.03300	1.68494
0.80	0.61939	- 1.02342	0.19545	1.74814
0.82	0.52370	- 0.99702	0.36346	1.80219
0.84	0.43126	- 0.95451	0.53615	1.84732
0.86	0.34363	- 0.89546	0.71270	1.88393
0.88	0.26237	- 0.81956	0.89234	1.91254
0.90	0.18910	- 0.72655	1.07433	1.93383
0.92	0.12545	- 0.61624	1.25802	1.94862
0.94	0.07306	- 0.48850	1.44284	1.95792
0.96	0.03358	- 0.34324	1.62832	1.96285
0.98	0.00867	- 0.18041	1.81412	1.96473
1.00	0.00000	0.00000	2.00000	1.96500

TABLE 1

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-CLAMPED BEAM
Second Mode

$\frac{x}{\ell}$	ϕ_2	$\phi_2' = \frac{1}{\beta_2}\frac{d\phi_2}{dx}$	$\phi_2'' = \frac{1}{\beta_2^2}\frac{d^2\phi_2}{dx^2}$	$\phi_2''' = \frac{1}{\beta_2^3}\frac{d^3\phi_2}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00155
0.02	0.02338	0.28944	1.68568	- 2.00031
0.04	0.08834	0.52955	1.37202	- 1.99205
0.06	0.18715	0.72055	1.06061	- 1.97080
0.08	0.31214	0.86296	0.75386	- 1,93186
0.10	0.45573	0.95776	0.45486	- 1.87176
0.12	0.61058	1.00644	0.16713	- 1.78813
0.14	0.76958	1.01105	- 0.10554	- 1.67975
0.16	0.92602	0.97427	- 0.35923	- 1.54652
0.18	1.07363	0.89940	- 0.59010	- 1.38933
0.20	1.20674	0.79030	- 0.79450	- 1.21002
0.22	1.32032	0.65138	- 0.96918	- 1.01127
0.24	1.41005	0.48755	- 1.11133	- 0.79651
0.26	1.47245	0.30410	- 1.21876	- 0.56977
0.28	1.50485	0.10660	- 1.28991	- 0.33555
0.30	1.50550	- 0.09916	- 1.32402	- 0.09872
0.32	1.47357	- 0.30736	- 1.32106	0.13566
0.34	1.40914	- 0.51224	- 1.28181	0.36246
0.36	1.31314	- 0.70819	- 1.20786	0.57665
0.38	1.18740	- 0.88997	- 1.10157	0.77340
0.40	1.03457	- 1.05271	- 0.96605	0.94823
0.42	0.85794	- 1.19209	- 0.80507	1.09714
0.44	0.66150	- 1.30448	- 0.62296	1.21670
0.46	0.44973	- 1.38693	- 0.42456	1.30414
0.48	0.22751	- 1.43728	- 0.21508	1.35744
0.50	0.00000	- 1.45420	0.00000	1.37532
0.52	- 0.22751	- 1.43728	0.21508	1.35744
0.54	- 0.44973	- 1.38693	0.42456	1.30414
0.56	- 0.66150	- 1.30448	0.62296	1.21670
0.58	- 0.85794	- 1.19209	0.80507	1.09714
0.60	- 1.03457	- 1.05271	0.96605	0.94823
0.62	- 1.18740	- 0.88997	1.10157	0.77340
0.64	- 1.31314	- 0.70819	1.20786	0.57665
0.66	- 1.40914	- 0.51224	1.28181	0.36246
0.68	- 1.47357	- 0.30736	1.32106	0.13566
0.70	- 1.50550	- 0.09916	1.32402	- 0.09872
0.72	- 1.50485	0.10660	1.28991	- 0.33555
0.74	- 1.47245	0.30410	1.21876	- 0.56977
0.76	- 1.41005	0.48755	1.11133	- 0.79651
0.78	- 1.32032	0.65138	0.96918	- 1.01127
0.80	- 1.20674	0.79030	0.79450	- 1.21002
0.82	- 1.07363	0.89940	0.59010	- 1.38933
0.84	- 0.92602	0.97427	0.35923	- 1.54652
0.86	- 0.76958	1.01105	0.10554	- 1.67975
0.88	- 0.61058	1.00644	- 0.16713	- 1.78813
0.90	- 0.45573	0.95776	- 0.45486	- 1.87176
0.92	- 0.31214	0.86296	- 0.75386	- 1.93186
0.94	- 0.18715	0.72055	- 1.06061	- 1.97080
0.96	- 0.08834	0.52955	- 1.37202	- 1.99205
0.98	- 0.02338	0.28944	- 1.68568	- 2.00031
1.00	0.00000	0.00000	- 2.00000	- 2.00155

TABLE 1

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-CLAMPED BEAM
Third Mode

$\frac{x}{l}$	ϕ_3	$\phi_3' = \frac{1}{\beta_3}\frac{d\phi_3}{dx}$	$\phi_3'' = \frac{1}{\beta_3^2}\frac{d^2\phi_3}{dx^2}$	$\phi_3''' = \frac{1}{\beta_3^3}\frac{d^3\phi_3}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 1.99993
0.02	0.04481	0.39147	1.56038	- 1.99658
0.04	0.16510	0.68646	1.12323	- 1.97469
0.06	0.33975	0.88609	0.69428	- 1.91998
0.08	0.54804	0.99303	0.28189	- 1.82280
0.10	0.77005	1.01202	- 0.10393	- 1.67795
0.12	0.98720	0.95006	- 0.45252	- 1.48447
0.14	1.18265	0.81649	- 0.75348	- 1.24535
0.16	1.34190	0.62285	- 0.99738	- 0.96698
0.18	1.45317	0.38256	- 1.17657	- 0.65867
0.20	1.50782	0.11050	- 1.28572	- 0.33199
0.22	1.50059	- 0.17759	- 1.32220	- 0.00005
0.24	1.42971	- 0.46573	- 1.28637	0.32333
0.26	1.29690	- 0.73833	- 1.18165	0.62425
0.28	1.10719	- 0.98087	- 1.01443	0.88956
0.30	0.86864	- 1.18057	- 0.79387	1.10762
0.32	0.59186	- 1.32694	- 0.53145	1.26880
0.34	0.28949	- 1.41222	- 0.24051	1.36606
0.36	- 0.02445	- 1.43171	0.06438	1.39529
0.38	- 0.33528	- 1.38399	0.36811	1.35554
0.40	- 0.62837	- 1.27099	0.65569	1.24912
0.42	- 0.88987	- 1.09782	0.91301	1.08148
0.44	- 1.10739	- 0.87257	1.12747	0.86096
0.46	- 1.27060	- 0.60586	1.28860	0.59842
0.48	- 1.37174	- 0.31031	1.38852	0.30669
0.50	- 1.40600	0.00000	1.42238	0.00000
0.52	- 1.37174	0.31031	1.38852	- 0.30669
0.54	- 1.27060	0.60586	1.28860	- 0.59842
0.56	- 1.10739	0.87257	1.12747	- 0.86096
0.58	- 0.88987	1.09782	0.91301	- 1.08148
0.60	- 0.62837	1.27099	0.65569	- 1.24912
0.62	- 0.33528	1.38399	0.36811	- 1.35554
0.64	- 0.02445	1.43171	0.06438	- 1.39529
0.66	0.28949	1.41222	- 0.24051	- 1.36606
0.68	0.59186	1.32694	- 0.53145	- 1.26880
0.70	0.86864	1.18057	- 0.79387	- 1.10762
0.72	1.10719	0.98087	- 1.01443	- 0.88956
0.74	1.29690	0.73833	- 1.18165	- 0.62425
0.76	1.42971	0.46573	- 1.28637	- 0.32333
0.78	1.50059	0.17759	- 1.32220	0.00005
0.80	1.50782	- 0.11050	- 1.28572	0.33199
0.82	1.45317	- 0.38256	- 1.17657	0.65867
0.84	1.34190	- 0.62285	- 0.99738	0.96698
0.86	1.18265	- 0.81649	- 0.75348	1.24535
0.88	0.98720	- 0.95006	- 0.45252	1.48447
0.90	0.77005	- 1.01202	- 0.10393	1.67795
0.92	0.54804	- 0.99303	0.28189	1.82280
0.94	0.33975	- 0.88609	0.69428	1.91998
0.96	0.16510	- 0.68646	1.12323	1.97469
0.98	0.04481	- 0.39147	1.56038	1.99658
1.00	0.00000	0.00000	2.00000	1.99993

TABLE 1

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-CLAMPED BEAM
Fourth Mode

$\frac{x}{\ell}$	ϕ_4	$\phi_4' = \frac{1}{\beta_4}\frac{d\phi_4}{dx}$	$\phi_4'' = \frac{1}{\beta_4^2}\frac{d^2\phi_4}{dx^2}$	$\phi_4''' = \frac{1}{\beta_4^3}\frac{d^3\phi_4}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00000
0.02	0.07241	0.48557	1.43502	- 1.99300
0.04	0.25958	0.81207	0.87658	- 1.94824
0.06	0.51697	0.98325	0.33937	- 1.83960
0.08	0.80177	1.00789	- 0.15633	- 1.65333
0.10	1.07449	0.90088	- 0.58802	- 1.38736
0.12	1.30078	0.68345	- 0.93412	- 1.05012
0.14	1.45308	0.38242	- 1.17673	- 0.65879
0.16	1.51208	0.02894	- 1.30380	- 0.23725
0.18	1.46765	- 0.34351	- 1.31068	0.18649
0.20	1.31923	- 0.70122	- 1.20092	0.58286
0.22	1.07550	- 1.01271	- 0.98634	0.92349
0.24	0.75348	- 1.25091	- 0.68630	1.18364
0.26	0.37700	- 1.39515	- 0.32640	1.34442
0.28	- 0.02537	- 1.43265	0.06348	1.39439
0.30	- 0.42268	- 1.35944	0.45136	1.33056
0.32	- 0.78413	- 1.18058	0.80569	1.15876
0.34	- 1.08159	- 0.90972	1.09776	0.89319
0.36	- 1.29186	- 0.56793	1.30395	0.55537
0.38	- 1.39858	- 0.18205	1.40755	0.17245
0.40	- 1.39351	0.21753	1.40010	- 0.22494
0.42	- 1.27726	0.59923	1.28198	- 0.60506
0.44	- 1.05920	0.93289	1.06244	- 0.93759
0.46	- 0.75676	1.19208	0.75879	- 1.19604
0.48	- 0.39407	1.35629	0.39504	- 1.35983
0.50	0.00000	1.41251	0.00000	- 1.41592
0.52	0.39407	1.35629	- 0.39504	- 1.35983
0.54	0.75676	1.19208	- 0.75879	- 1.19604
0.56	1.05920	0.93289	- 1.06244	- 0.93759
0.58	1.27726	0.59923	- 1.28198	- 0.60506
0.60	1.39351	0.21753	- 1.40010	- 0.22494
0.62	1.39858	- 0.18205	- 1.40755	0.17245
0.64	1.29186	- 0.56793	- 1.30395	0.55537
0.66	1.08159	- 0.90972	- 1.09776	0.89319
0.68	0.78413	- 1.18058	- 0.80569	1.15876
0.70	0.42268	- 1.35944	- 0.45136	1.33056
0.72	0.02537	- 1.43265	- 0.06348	1.39439
0.74	- 0.37700	- 1.39515	0.32640	1.34442
0.76	- 0.75348	- 1.25091	0.68630	1.18364
0.78	- 1.07550	- 1.01271	0.98634	0.92349
0.80	- 1.31923	- 0.70122	1.20092	0.58286
0.82	- 1.46765	- 0.34351	1.31068	0.18649
0.84	- 1.51208	0.02894	1.30380	- 0.23725
0.86	- 1.45308	0.38242	1.17673	- 0.65879
0.88	- 1.30078	0.68345	0.93412	- 1.05012
0.90	- 1.07449	0.90088	0.58802	- 1.38736
0.92	- 0.80177	1.00789	0.15633	- 1.65333
0.94	- 0.51697	0.98325	- 0.33937	- 1.83960
0.96	- 0.25958	0.81207	- 0.87658	- 1.94824
0.98	- 0.07241	0.48557	- 1.43502	- 1.99300
1.00	0.00000	0.00000	- 2.00000	- 2.00000

TABLE 1

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-CLAMPED BEAM
Fifth Mode

$\frac{x}{\ell}$	ϕ_5	$\phi_5' = \frac{1}{\beta_5}\frac{d\phi_5}{dx}$	$\phi_5'' = \frac{1}{\beta_5^2}\frac{d^2\phi_5}{dx^2}$	$\phi_5''' = \frac{1}{\beta_5^3}\frac{d^3\phi_5}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00000
0.02	0.10567	0.57181	1.30996	- 1.98743
0.04	0.36791	0.90694	0.63409	- 1.90894
0.06	0.70632	1.01517	0.00291	- 1.72440
0.08	1.04591	0.91867	- 0.54391	- 1.42067
0.10	1.32178	0.65359	- 0.96646	- 1.00891
0.12	1.48381	0.26880	- 1.23231	- 0.52030
0.14	1.50043	- 0.17781	- 1.32242	- 0.00021
0.16	1.36090	- 0.62465	- 1.23490	0.49865
0.18	1.07551	- 1.01269	- 0.98632	0.92351
0.20	0.67360	- 1.29164	- 0.61048	1.22851
0.22	0.09959	- 1.42540	- 0.15491	1.38072
0.24	- 0.29269	- 1.39597	0.32432	1.36434
0.26	- 0.74658	- 1.20525	0.76897	1.18287
0.28	- 1.10952	- 0.87470	1.12538	0.85886
0.30	- 1.33938	- 0.44262	1.35061	0.43141
0.32	- 1.40954	0.04046	1.41749	- 0.04838
0.34	- 1.31208	0.51781	1.31772	- 0.52341
0.36	- 1.05881	0.93326	1.06282	- 0.93721
0.38	- 0.67987	1.23790	0.68273	- 1.24067
0.40	- 0.22021	1.39584	0.22226	- 1.39777
0.42	0.26575	1.38850	- 0.26425	- 1.38983
0.44	0.72046	1.21684	- 0.71933	- 1.21771
0.46	1.09011	0.90119	- 1.08923	- 0.90172
0.48	1.33098	0.47892	- 1.33023	- 0.47917
0.50	1.41457	0.00000	- 1.41386	0.00000
0.52	1.33098	- 0.47892	- 1.33023	0.47917
0.54	1.09011	- 0.90119	- 1.08923	0.90172
0.56	0.72046	- 1.21684	- 0.71933	1.21771
0.58	0.26575	- 1.38850	- 0.26425	1.38983
0.60	- 0.22021	- 1.39584	0.22226	1.39777
0.62	- 0.67987	- 1.23790	0.68273	1.24067
0.64	- 1.05881	- 0.93326	1.06282	0.93721
0.66	- 1.31208	- 0.51781	1.31772	0.52341
0.68	- 1.40954	- 0.04046	1.41749	0.04838
0.70	- 1.33938	0.44262	1.35061	- 0.43141
0.72	- 1.10952	0.87470	1.12538	- 0.85886
0.74	- 0.74658	1.20525	0.76897	- 1.18287
0.76	- 0.29269	1.39597	0.32432	- 1.36434
0.78	0.19959	1.42540	- 0.15491	- 1.38072
0.80	0.67360	1.29164	- 0.61048	- 1.22851
0.82	1.07551	1.01269	- 0.98632	- 0.92351
0.84	1.36090	0.62465	- 1.23490	- 0.49865
0.86	1.50043	0.17781	- 1.32242	0.00021
0.88	1.48381	- 0.26880	- 1.23231	0.52030
0.90	1.32178	- 0.65359	- 0.96646	1.00891
0.92	1.04591	- 0.91867	- 0.54391	1.42067
0.94	0.70632	- 1.01517	0.00291	1.72440
0.96	0.36791	- 0.90694	0.63409	1.90894
0.98	0.10567	- 0.57181	1.30996	1.98743
1.00	0.00000	0.00000	2.00000	2.00000

Data for Clamped-Free Beam

Characteristic function and its derivatives

$$\phi_n = \operatorname{Cosh}\beta_n x - \cos\beta_n x - \alpha_n (\operatorname{Sinh}\beta_n x - \sin\beta_n x)$$

$$\frac{1}{\beta_n}\frac{d\phi_n}{dx} = \phi_n' = \operatorname{Sinh}\beta_n x + \sin\beta_n x - \alpha_n (\operatorname{Cosh}\beta_n x - \cos\beta_n x)$$

$$\frac{1}{\beta_n^2}\frac{d^2\phi_n}{dx^2} = \phi_n'' = \operatorname{Cosh}\beta_n x + \cos\beta_n x - \alpha_n (\operatorname{Sinh}\beta_n x + \sin\beta_n x)$$

$$\frac{1}{\beta_n^3}\frac{d^3\phi_n}{dx^3} = \phi_n''' = \operatorname{Sinh}\beta_n x - \sin\beta_n x - \alpha_n (\operatorname{Cosh}\beta_n x + \cos\beta_n x)$$

Boundary values

$$\phi_n(0) = \phi_n'(0) = 0$$

$$\phi_n''(\ell) = \phi_n'''(\ell) = 0$$

Values of α_n and $\beta_n\ell$ and various powers

n	$\beta_n\ell$	$(\beta_n\ell)^2$	$(\beta_n\ell)^3$	$(\beta_n\ell)^4$
1	1.8751 041	3.5160 154	6.5928 95	12.3623 64
2	4.6940 9113	22.0344 92	103.4319 1	485.5188
3	7.8547 5743	61.6972 14	484.6166 5	3 806.5462
4	10.9955 4074	120.9019 16	1329.3819 4	14 617.2733
5	14.1371 6839	199.8595 30	2825.4478 3	39 943.8318

n	α_n	$\frac{\omega_n}{\omega_1} = \frac{\beta_n^2}{\beta_1^2}$	$\frac{\omega_n^2}{\omega_1^2} = \frac{\beta_n^4}{\beta_1^4}$
1	0.7340 955	1.0	1.0
2	1.0184 6644	6.2668 928	39.2739 45
3	0.9992 2450	17.5474 81	307.9141 0
4	1.0000 3355 3	34.3860 59	1182.4011
5	0.9999 9855 01	56.8426 21	3231.0835

For $n > 5$

$\beta_n\ell \doteq (2n-1)\pi/2$

$\alpha_n \doteq 1.0$

TABLE 2

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-FREE BEAM
First Mode

$\frac{x}{\ell}$	ϕ_1	$\phi_1' = \frac{1}{\beta_1}\frac{d\phi_1}{dx}$	$\phi_1'' = \frac{1}{\beta_1^2}\frac{d^2\phi_1}{dx^2}$	$\phi_1''' = \frac{1}{\beta_1^3}\frac{d^3\phi_1}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 1.46819
0.02	0.00139	0.07397	1.94494	- 1.46817
0.04	0.00552	0.14588	1.88988	- 1.46805
0.06	0.01231	0.21572	1.83483	- 1.46773
0.08	0.02168	0.28350	1.77980	- 1.46710
0.10	0.03355	0.34921	1.72480	- 1.46607
0.12	0.04784	0.41286	1.66985	- 1.46455
0.14	0.06449	0.47446	1.61496	- 1.46245
0.16	0.08340	0.53400	1.56016	- 1.45968
0.18	0.10452	0.59148	1.50549	- 1.45617
0.20	0.12774	0.64692	1.45096	- 1.45182
0.22	0.15301	0.70031	1.39660	- 1.44656
0.24	0.18024	0.75167	1.34247	- 1.44032
0.26	0.20936	0.80100	1.28859	- 1.43302
0.28	0.24030	0.84832	1.23500	- 1.42459
0.30	0.27297	0.89364	1.18175	- 1.41497
0.32	0.30730	0.93696	1.12889	- 1.40410
0.34	0.34322	0.97831	1.07646	- 1.39191
0.36	0.38065	1.01771	1.02451	- 1.37834
0.38	0.41952	1.05516	0.97309	- 1.36334
0.40	0.45977	1.09070	0.92227	- 1.34685
0.42	0.50131	1.12435	0.87209	- 1.32884
0.44	0.54408	1.15612	0.82262	- 1.30924
0.46	0.58800	1.18606	0.77392	- 1.28801
0.48	0.63301	1.21418	0.72603	- 1.26512
0.50	0.67905	1.24052	0.67905	- 1.24052
0.52	0.72603	1.26512	0.63301	- 1.21418
0.54	0.77392	1.28801	0.58800	- 1.18606
0.56	0.82262	1.30924	0.54408	- 1.15612
0.58	0.87209	1.32884	0.50131	- 1.12435
0.60	0.92227	1.34685	0.45977	- 1.09070
0.62	0.97309	1.36334	0.41952	- 1.05516
0.64	1.02451	1.37834	0.38065	- 1.01771
0.66	1.07646	1.39191	0.34322	- 0.97831
0.68	1.12889	1.40410	0.30730	- 0.93696
0.70	1.18175	1.41497	0.27297	- 0.89364
0.72	1.23500	1.42459	0.24030	- 0.84832
0.74	1.28859	1.43302	0.20936	- 0.80100
0.76	1.34247	1.44032	0.18024	- 0.75167
0.78	1.39660	1.44656	0.15301	- 0.70031
0.80	1.45096	1.45182	0.12774	- 0.64692
0.82	1.50549	1.45617	0.10452	- 0.59148
0.84	1.56016	1.45968	0.08340	- 0.53400
0.86	1.61496	1.46245	0.06449	- 0.47446
0.88	1.66985	1.46455	0.04784	- 0.41286
0.90	1.72480	1.46607	0.03355	- 0.34921
0.92	1.77980	1.46710	0.02168	- 0.28350
0.94	1.83483	1.46773	0.01231	- 0.21572
0.96	1.88988	1.46805	0.00552	- 0.14588
0.98	1.94494	1.46817	0.00139	- 0.07397
1.00	2.00000	1.46819	0.00000	0.00000

TABLE 2

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-FREE BEAM
Second Mode

$\frac{x}{\ell}$	ϕ_2	$\phi_2' = \frac{1}{\beta_2}\frac{d\phi_2}{dx}$	$\phi_2'' = \frac{1}{\beta_2^2}\frac{d^2\phi_2}{dx^2}$	$\phi_2''' = \frac{1}{\beta_2^3}\frac{d^3\phi_2}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.03693
0.02	0.00853	0.17879	1.80877	- 2.03666
0.04	0.03301	0.33962	1.61764	- 2.03483
0.06	0.07174	0.48253	1.42680	- 2.03002
0.08	0.12305	0.60754	1.23660	- 2.02097
0.10	0.18526	0.71475	1.04750	- 2.00658
0.12	0.25670	0.80428	0.86004	- 1.98590
0.14	0.33573	0.87631	0.67484	- 1.95814
0.16	0.42070	0.93108	0.49261	- 1.92267
0.18	0.51002	0.96892	0.31409	- 1.87901
0.20	0.60211	0.99020	0.14007	- 1.82682
0.22	0.69544	0.99539	- 0.02865	- 1.76592
0.24	0.78852	0.98502	- 0.19123	- 1.69625
0.26	0.87992	0.95970	- 0.34687	- 1.61791
0.28	0.96827	0.92013	- 0.49475	- 1.53113
0.30	1.05227	0.86707	- 0.63410	- 1.43624
0.32	1.13068	0.80136	- 0.76419	- 1.33373
0.34	1.20236	0.72389	- 0.88431	- 1.22416
0.36	1.26626	0.63565	- 0.99384	- 1.10821
0.38	1.32141	0.53764	- 1.09222	- 0.98667
0.40	1.36694	0.43094	- 1.17895	- 0.86040
0.42	1.40209	0.31665	- 1.25365	- 0.73034
0.44	1.42619	0.19593	- 1.31600	- 0.59748
0.46	1.43871	0.06995	- 1.36578	- 0.46291
0.48	1.43920	- 0.06012	- 1.40289	- 0.32772
0.50	1.42733	- 0.19307	- 1.42733	- 0.19307
0.52	1.40289	- 0.32772	- 1.43920	- 0.06012
0.54	1.36578	- 0.46291	- 1.43871	0.06995
0.56	1.31600	- 0.59748	- 1.42619	0.19593
0.58	1.25365	- 0.73034	- 1.40209	0.31665
0.60	1.17895	- 0.86040	- 1.36694	0.43094
0.62	1.09222	- 0.98667	- 1.32141	0.53764
0.64	0.99384	- 1.10821	- 1.26626	0.63565
0.66	0.88431	- 1.22416	- 1.20236	0.72389
0.68	0.76419	- 1.33373	- 1.13068	0.80136
0.70	0.63410	- 1.43624	- 1.05227	0.86707
0.72	0.49475	- 1.53113	- 0.96827	0.92013
0.74	0.34687	- 1.61791	- 0.87992	0.95970
0.76	0.19123	- 1.69625	- 0.78852	0.98502
0.78	0.02865	- 1.76592	- 0.69544	0.99539
0.80	- 0.14007	- 1.82682	- 0.60211	0.99020
0.82	- 0.31409	- 1.87901	- 0.51002	0.96892
0.84	- 0.49261	- 1.92267	- 0.42070	0.93108
0.86	- 0.67484	- 1.95814	- 0.33573	0.87631
0.88	- 0.86004	- 1.98590	- 0.25670	0.80428
0.90	- 1.04750	- 2.00658	- 0.18526	0.71475
0.92	- 1.23660	- 2.02097	- 0.12305	0.60754
0.94	- 1.42680	- 2.03002	- 0.07174	0.48253
0.96	- 1.61764	- 2.03483	- 0.03301	0.33962
0.98	- 1.80877	- 2.03666	- 0.00853	0.17879
1.00	- 2.00000	- 2.03693	0.00000	0.00000

TABLE 2

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-FREE BEAM
Third Mode

$\frac{x}{l}$	ϕ_3	$\phi_3' = \frac{1}{\beta_3}\frac{d\phi_3}{dx}$	$\phi_3'' = \frac{1}{\beta_3^2}\frac{d^2\phi_3}{dx^2}$	$\phi_3''' = \frac{1}{\beta_3^3}\frac{d^3\phi_3}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 1.99845
0.02	0.02339	0.28953	1.68610	- 1.99721
0.04	0.08839	0.52979	1.37287	- 1.98892
0.06	0.18727	0.72099	1.06189	- 1.96766
0.08	0.31238	0.86367	0.75558	- 1.92871
0.10	0.45614	0.95879	0.45702	- 1.86854
0.12	0.61120	1.00785	0.16974	- 1.78480
0.14	0.77049	1.01291	- 0.10245	- 1.67629
0.16	0.92728	0.97665	- 0.35563	- 1.54286
0.18	1.07535	0.90237	- 0.58594	- 1.38540
0.20	1.20901	0.79394	- 0.78975	- 1.20575
0.22	1.32324	0.65580	- 0.96375	- 1.00656
0.24	1.41376	0.49285	- 1.10515	- 0.79124
0.26	1.47707	0.31040	- 1.21172	- 0.56380
0.28	1.51056	0.11405	- 1.28189	- 0.32872
0.30	1.51248	- 0.09041	- 1.31485	- 0.09085
0.32	1.48203	- 0.29711	- 1.31055	0.14479
0.34	1.41931	- 0.50026	- 1.26974	0.37310
0.36	1.32534	- 0.69422	- 1.19398	0.58908
0.38	1.20196	- 0.87368	- 1.08556	0.78797
0.40	1.05185	- 1.03374	- 0.94753	0.96533
0.42	0.87841	- 1.17003	- 0.78359	1.11723
0.44	0.68568	- 1.27881	- 0.59802	1.24030
0.46	0.47822	- 1.35704	- 0.39555	1.33188
0.48	0.26103	- 1.40247	- 0.18130	1.39004
0.50	0.03937	- 1.41366	0.03937	1.41366
0.52	- 0.18130	- 1.39004	0.26103	1.40247
0.54	- 0.39555	- 1.33188	0.47822	1.35704
0.56	- 0.59802	- 1.24030	0.68568	1.27881
0.58	- 0.78359	- 1.11723	0.87841	1.17003
0.60	- 0.94753	- 0.96533	1.05185	1.03374
0.62	- 1.08556	- 0.78797	1.20196	0.87368
0.64	- 1.19398	- 0.58908	1.32534	0.69422
0.66	- 1.26974	- 0.37310	1.41931	0.50026
0.68	- 1.31055	- 0.14479	1.48203	0.29711
0.70	- 1.31485	0.09085	1.51248	0.09041
0.72	- 1.28189	0.32872	1.51056	- 0.11405
0.74	- 1.21172	0.56380	1.47707	- 0.31040
0.76	- 1.10515	0.79124	1.41376	- 0.49285
0.78	- 0.96375	1.00656	1.32324	- 0.65580
0.80	- 0.78975	1.20575	1.20901	- 0.79394
0.82	- 0.58594	1.38540	1.07535	- 0.90237
0.84	- 0.35563	1.54236	0.92728	- 0.97665
0.86	- 0.10245	1.67629	0.77049	- 1.01291
0.88	0.16974	1.78480	0.61120	- 1.00785
0.90	0.45702	1.86854	0.45614	- 0.95879
0.92	0.75558	1.92871	0.31238	- 0.86367
0.94	1.06189	1.96766	0.18727	- 0.72099
0.96	1.37287	1.98892	0.08829	- 0.52979
0.98	1.68610	1.99721	0.02339	- 0.28953
1.00	2.00000	1.99845	0.00000	0.00000

TABLE 2

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-FREE BEAM
Fourth Mode

$\frac{x}{\ell}$	ϕ_4	$\phi_4' = \frac{1}{\beta_4}\frac{d\phi_4}{dx}$	$\phi_4'' = \frac{1}{\beta_4^2}\frac{d^2\phi_4}{dx^2}$	$\phi_4''' = \frac{1}{\beta_4^3}\frac{d^3\phi_4}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00007
0.02	0.04482	0.39147	1.56035	- 1.99672
0.04	0.16510	0.68645	1.12317	- 1.97482
0.06	0.33974	0.88606	0.69420	- 1.92012
0.08	0.54801	0.99298	0.28179	- 1.82294
0.10	0.77002	1.01194	- 0.10407	- 1.67809
0.12	0.98714	0.94994	- 0.45270	- 1.48463
0.14	1.18256	0.81633	- 0.75368	- 1.24552
0.16	1.34177	0.62264	- 0.99762	- 0.96717
0.18	1.45299	0.38230	- 1.17687	- 0.65891
0.20	1.50758	0.11017	- 1.28608	- 0.33228
0.22	1.50027	- 0.17801	- 1.32262	- 0.00038
0.24	1.42928	- 0.46624	- 1.28688	0.32290
0.26	1.29634	- 0.73895	- 1.18226	0.62370
0.28	1.10648	- 0.98164	- 1.01518	0.88888
0.30	0.86774	- 1.18154	- 0.79478	1.10676
0.32	0.59073	- 1.32813	- 0.53258	1.26772
0.34	0.28808	- 1.41368	- 0.24191	1.36469
0.36	- 0.02621	- 1.43351	0.06264	1.39357
0.38	- 0.33748	- 1.38622	0.36594	1.35339
0.40	- 0.63112	- 1.27376	0.65299	1.24643
0.42	- 0.89330	- 1.10126	0.90964	1.07812
0.44	- 1.11166	- 0.87683	1.12327	0.85675
0.46	- 1.27592	- 0.61115	1.28336	0.59315
0.48	- 1.37836	- 0.31690	1.38199	0.30011
0.50	- 1.41424	- 0.00819	1.41424	- 0.00819
0.52	- 1.38199	0.30012	1.37836	- 0.31690
0.54	- 1.28336	0.59316	1.27592	- 0.61115
0.56	- 1.12327	0.85675	1.11166	- 0.87684
0.58	- 0.90964	1.07812	0.89330	- 1.10126
0.60	- 0.65299	1.24643	0.63112	- 1.27376
0.62	- 0.36594	1.35339	0.33748	- 1.38622
0.64	- 0.06264	1.39357	0.02621	- 1.43351
0.66	0.24191	1.36469	- 0.28808	- 1.41368
0.68	0.53258	1.26772	- 0.59073	- 1.32813
0.70	0.79478	1.10676	- 0.86774	- 1.18153
0.72	1.01518	0.88888	- 1.10648	- 0.98164
0.74	1.18226	0.62370	- 1.29634	- 0.73895
0.76	1.28688	0.32290	- 1.42928	- 0.46624
0.78	1.32262	- 0.00039	- 1.50027	- 0.17801
0.80	1.28608	- 0.33228	- 1.50758	0.11017
0.82	1.17687	- 0.65890	- 1.45299	0.38230
0.84	0.99762	- 0.96717	- 1.34177	0.62264
0.86	0.75368	- 1.24552	- 1.18256	0.81633
0.88	0.45270	- 1.48463	- 0.98714	0.94994
0.90	0.10407	- 1.67809	- 0.77002	1.01194
0.92	- 0.28179	- 1.82294	- 0.54801	0.99298
0.94	- 0.69420	- 1.92012	- 0.33974	0.88606
0.96	- 1.12317	- 1.97482	- 0.16510	0.68645
0.98	- 1.56035	- 1.99672	- 0.04482	0.39147
1.00	- 2.00000	- 2.00007	0.00000	0.00000

TABLE 2

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-FREE BEAM
Fifth Mode

$\frac{x}{\ell}$	ϕ_5	$\phi_5' = \frac{1}{\beta_5}\frac{d\phi_5}{dx}$	$\phi_5'' = \frac{1}{\beta_5^2}\frac{d^2\phi_5}{dx^2}$	$\phi_5''' = \frac{1}{\beta_5^3}\frac{d^3\phi_5}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00000
0.02	0.07241	0.48557	1.43502	- 1.99300
0.04	0.25958	0.81207	0.87658	- 1.94824
0.06	0.51697	0.98325	0.33937	- 1.83959
0.08	0.80177	1.00789	- 0.15633	- 1.65332
0.10	1.07449	0.90089	- 0.58801	- 1.38736
0.12	1.30078	0.68346	- 0.93411	- 1.05011
0.14	1.45309	0.38243	- 1.17672	- 0.65878
0.16	1.51209	0.02895	- 1.30378	- 0.23723
0.18	1.46767	- 0.34348	- 1.31066	0.18651
0.20	1.31925	- 0.70119	- 1.20090	0.58289
0.22	1.07553	- 1.01267	- 0.98631	0.92352
0.24	0.75353	- 1.25086	- 0.68626	1.18368
0.26	0.37706	- 1.39509	- 0.32634	1.34448
0.28	- 0.02529	- 1.43257	0.06355	1.39446
0.30	- 0.42257	- 1.35934	0.45146	1.33065
0.32	- 0.78399	- 1.18045	0.80582	1.15889
0.34	- 1.08140	- 0.90954	1.09793	0.89337
0.36	- 1.29162	- 0.56770	1.30418	0.55561
0.38	- 1.39826	- 0.18174	1.40786	0.17276
0.40	- 1.39310	0.21794	1.40051	- 0.22452
0.42	- 1.27670	0.59978	1.28253	- 0.60450
0.44	- 1.05846	0.93361	1.06317	- 0.93686
0.46	- 0.75579	1.19304	0.75976	- 1.19508
0.48	- 0.39278	1.35757	0.39632	- 1.35855
0.50	0.00170	1.41421	0.00170	- 1.41421
0.52	0.39632	1.35855	- 0.39278	- 1.35757
0.54	0.75976	1.19508	- 0.75579	- 1.19304
0.56	1.06317	0.93686	- 1.05846	- 0.93361
0.58	1.28253	0.60450	- 1.27670	- 0.59978
0.60	1.40051	0.22452	- 1.39310	- 0.21794
0.62	1.40786	- 0.17276	- 1.39826	0.18174
0.64	1.30418	- 0.55561	- 1.29162	0.56770
0.66	1.09793	- 0.89337	- 1.08140	0.90954
0.68	0.80582	- 1.15889	- 0.78399	1.18045
0.70	0.45146	- 1.33065	- 0.42257	1.35934
0.72	0.06355	- 1.39446	- 0.02529	1.43257
0.74	- 0.32634	- 1.34448	0.37706	1.39509
0.76	- 0.68626	- 1.18368	0.75353	1.25086
0.78	- 0.98631	- 0.92352	1.07553	1.01267
0.80	- 1.20090	- 0.58289	1.31925	0.70119
0.82	- 1.31066	- 0.18651	1.46767	0.34348
0.84	- 1.30378	0.23723	1.51209	- 0.02895
0.86	- 1.17672	0.65878	1.45309	- 0.38243
0.88	- 0.93411	1.05011	1.30078	- 0.68346
0.90	- 0.58801	1.38736	1.07449	- 0.90089
0.92	- 0.15633	1.65332	0.80177	- 1.00789
0.94	0.33937	1.83959	0.51697	- 0.98325
0.96	0.87658	1.94824	0.25958	- 0.81207
0.98	1.43502	1.99300	0.07241	- 0.48557
1.00	2.00000	2.00000	0.00000	0.00000

Data for Clamped-Supported Beam and Free-Supported Beam

A. CLAMPED-SUPPORTED BEAM

Characteristic function and its derivatives

$$\phi_n = \operatorname{Cosh}\beta_n x - \cos\beta_n x - \alpha_n (\operatorname{Sinh}\beta_n x - \sin\beta_n x)$$

$$\frac{1}{\beta_n}\frac{d\phi_n}{dx} = \phi_n' = \operatorname{Sinh}\beta_n x + \sin\beta_n x - \alpha_n (\operatorname{Cosh}\beta_n x - \cos\beta_n x)$$

$$\frac{1}{\beta_n^2}\frac{d^2\phi_n}{dx^2} = \phi_n'' = \operatorname{Cosh}\beta_n x + \cos\beta_n x - \alpha_n (\operatorname{Sinh}\beta_n x + \sin\beta_n x)$$

$$\frac{1}{\beta_n^3}\frac{d^3\phi_n}{dx^3} = \phi_n''' = \operatorname{Sinh}\beta_n x - \sin\beta_n x - \alpha_n (\operatorname{Cosh}\beta_n x + \cos\beta_n x)$$

Boundary values

$$\phi_n(0) = \phi_n'(0) = 0$$

$$\phi_n(\ell) = \phi_n''(\ell) = 0$$

Values of α_n and $\beta_n\ell$ and various powers

n	$\beta_n\ell$	$(\beta_n\ell)^2$	$(\beta_n\ell)^3$	$(\beta_n\ell)^4$
1	3.9266 023	15.4182 06	60.5411 6	237.7210 7
2	7.0685 8275	49.9648 62	353.1807 6	2496.4874
3	10.2101 7613	104.2476 97	1064.3873 4	10 867.5822
4	13.3517 6878	178.2697 30	2380.2162 1	31 780.0965
5	16.4933 6143	272.0309 71	4486.7051 3	74 000.8494

n	α_n	$\frac{\omega_n}{\omega_1} = \frac{\beta_n^2}{\beta_1^2}$	$\frac{\omega_n^2}{\omega_1^2} = \frac{\beta_n^4}{\beta_1^4}$
1	1.0007 7730	1.0	1.0
2	1.0000 0144	3.2406 405	10.5017 50
3	1.0000 0000	6.7613 377	45.7156 88
4	1.0000 0000	11.5622 876	133.6864 9
5	1.0000 0000	17.6434 908	311.2927 7

For $n > 5$

$\beta_n\ell \doteq (4n+1)\pi/4$

$\alpha_n \doteq 1.0$

B. FREE-SUPPORTED BEAM

Characteristic function

$$\phi_n = \text{Cosh}\,\beta_n x + \cos\beta_n x - \alpha_n(\text{Sinh}\,\beta_n x + \sin\beta_n x)$$

where α_n and β_n are the same as for a clamped-supported beam.

The characteristic function for a free-supported beam is the same as the second derivative of a clamped-supported beam, that is

$$\phi_n \text{ (free-supported)} = \phi_n'' \text{ (clamped-supported)}$$
$$\phi_n' \text{ (free-supported)} = \phi_n''' \text{ (clamped-supported)}$$
$$\phi_n'' \text{ (free-supported)} = \phi_n \text{ (clamped-supported)}$$
$$\phi_n''' \text{ (free-supported)} = \phi_n' \text{ (clamped-supported)}$$

Boundary values

$$\phi_n''(0) = \phi_n'''(0) = 0$$
$$\phi_n(\ell) = \phi_n''(\ell) = 0$$

TABLE 3

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-SUPPORTED BEAM
First Mode

$\frac{x}{\ell}$	ϕ_1	$\phi_1' = \frac{1}{\beta_1}\frac{d\phi_1}{dx}$	$\phi_1'' = \frac{1}{\beta_1^2}\frac{d^2\phi_1}{dx^2}$	$\phi_1''' = \frac{1}{\beta_1^3}\frac{d^3\phi_1}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00155
0.02	0.00600	0.15089	1.84282	- 2.00140
0.04	0.02338	0.28944	1.68568	- 2.00031
0.06	0.05114	0.41566	1.52869	- 1.99745
0.08	0.08834	0.52955	1.37202	- 1.99203
0.10	0.13400	0.63116	1.21590	- 1.98336
0.12	0.18715	0.72055	1.06060	- 1.97079
0.14	0.24685	0.79778	0.90647	- 1.95379
0.16	0.31214	0.86296	0.75386	- 1.93187
0.18	0.38208	0.91623	0.60318	- 1.90464
0.20	0.45574	0.95776	0.45486	- 1.87177
0.22	0.53221	0.98775	0.30935	- 1.83299
0.24	0.61058	1.00643	0.16712	- 1.78812
0.26	0.68999	1.01410	0.02866	- 1.73706
0.28	0.76958	1.01105	- 0.10554	- 1.67975
0.30	0.84852	0.99764	- 0.23500	- 1.61620
0.32	0.92601	0.97427	- 0.35923	- 1.54652
0.34	1.00129	0.94137	- 0.47775	- 1.47082
0.36	1.07363	0.89940	- 0.59009	- 1.38932
0.38	1.14233	0.84886	- 0.69582	- 1.30229
0.40	1.20675	0.79029	- 0.79450	- 1.21002
0.42	1.26626	0.72427	- 0.88574	- 1.11288
0.44	1.32032	0.65138	- 0.96918	- 1.01128
0.46	1.36841	0.57226	- 1.04447	- 0.90566
0.48	1.41006	0.48755	- 1.11133	- 0.79652
0.50	1.44486	0.39794	- 1.16950	- 0.68437
0.52	1.47245	0.30410	- 1.21875	- 0.56977
0.54	1.49253	0.20675	- 1.25894	- 0.45330
0.56	1.50485	0.10661	- 1.28992	- 0.33555
0.58	1.50922	0.00440	- 1.31162	- 0.21715
0.60	1.50550	- 0.09916	- 1.32402	- 0.09872
0.62	1.49363	- 0.20332	- 1.32714	0.01910
0.64	1.47357	- 0.30736	- 1.32106	0.13566
0.66	1.44537	- 0.41057	- 1.30588	0.25033
0.68	1.40913	- 0.51224	- 1.28180	0.36247
0.70	1.36498	- 0.61167	- 1.24904	0.47145
0.72	1.31313	- 0.70820	- 1.20786	0.57666
0.74	1.25384	- 0.80117	- 1.15858	0.67750
0.76	1.18741	- 0.88996	- 1.10157	0.77340
0.78	1.11418	- 0.97400	- 1.03725	0.86382
0.80	1.03457	- 1.05270	- 0.96606	0.94823
0.82	0.94899	- 1.12556	- 0.88849	1.02616
0.84	0.85795	- 1.19210	- 0.80507	1.09714
0.86	0.76194	- 1.25187	- 0.71636	1.16078
0.88	0.66151	- 1.30448	- 0.62295	1.21670
0.90	0.55724	- 1.34960	- 0.52547	1.26458
0.92	0.44974	- 1.38693	- 0.42455	1.30414
0.94	0.33962	- 1.41621	- 0.32086	1.33515
0.96	0.22752	- 1.43727	- 0.21507	1.35743
0.98	0.11410	- 1.44996	- 0.10789	1.37085
1.00	0.00000	- 1.45420	0.00000	1.37533

TABLE 3

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-SUPPORTED BEAM
Second Mode

$\frac{x}{\ell}$	ϕ_2	$\phi_2' = \frac{1}{\beta_2}\frac{d\phi_2}{dx}$	$\phi_2'' = \frac{1}{\beta_2^2}\frac{d^2\phi_2}{dx^2}$	$\phi_2''' = \frac{1}{\beta_2^3}\frac{d^3\phi_2}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00000
0.02	0.01904	0.26276	1.71729	- 1.99910
0.04	0.07241	0.48557	1.43502	- 1.99300
0.06	0.15446	0.66857	1.15424	- 1.97727
0.08	0.25958	0.81207	0.87658	- 1.94824
0.10	0.38223	0.91666	0.60415	- 1.90305
0.12	0.51697	0.98325	0.33937	- 1.83960
0.14	0.65851	1.01310	0.08494	- 1.75656
0.16	0.80176	1.00789	- 0.15633	- 1.65333
0.18	0.94192	0.96966	- 0.38158	- 1.53001
0.20	1.07449	0.90088	- 0.58802	- 1.38736
0.22	1.19534	0.80441	- 0.77300	- 1.22676
0.24	1.30078	0.68345	- 0.93412	- 1.05012
0.26	1.38759	0.54152	- 1.06927	- 0.85985
0.28	1.45308	0.38242	- 1.17673	- 0.65879
0.30	1.49510	0.21017	- 1.25518	- 0.45011
0.32	1.51208	0.02894	- 1.30380	- 0.23724
0.34	1.50305	- 0.15704	- 1.32224	- 0.02381
0.36	1.46765	- 0.34350	- 1.31068	0.18649
0.38	1.40611	- 0.52625	- 1.26983	0.38993
0.40	1.31923	- 0.70122	- 1.20092	0.58286
0.42	1.20839	- 0.86456	- 1.10569	0.76180
0.44	1.07550	- 1.01270	- 0.98634	0.92349
0.46	0.92292	- 1.14243	- 0.84553	1.06496
0.48	0.75348	- 1.25090	- 0.68631	1.18364
0.50	0.57035	- 1.33577	- 0.51204	1.27736
0.52	0.37700	- 1.39515	- 0.32640	1.34442
0.54	0.17715	- 1.42770	- 0.13323	1.38365
0.56	- 0.02536	- 1.43265	0.06348	1.39438
0.58	- 0.22661	- 1.40978	0.25968	1.37654
0.60	- 0.42268	- 1.35944	0.45136	1.33056
0.62	- 0.60973	- 1.28256	0.63460	1.25745
0.64	- 0.78413	- 1.18058	0.80569	1.15876
0.66	- 0.94244	- 1.05549	0.96112	1.03650
0.68	- 1.08158	- 0.90972	1.09776	0.89319
0.70	- 1.19882	- 0.74612	1.21281	0.73172
0.72	- 1.29186	- 0.56793	1.30395	0.55537
0.74	- 1.35888	- 0.37866	1.36930	0.36769
0.76	- 1.39858	- 0.18205	1.40755	0.17245
0.78	- 1.41019	0.01800	1.41789	- 0.02643
0.80	- 1.39351	0.21752	1.40010	- 0.22494
0.82	- 1.34890	0.41256	1.35450	- 0.41912
0.84	- 1.27726	0.59923	1.28198	- 0.60506
0.86	- 1.18004	0.77383	1.18399	- 0.77904
0.88	- 1.05919	0.93288	1.06244	- 0.93759
0.90	- 0.91715	1.07323	0.91976	- 1.07752
0.92	- 0.75676	1.19208	0.75879	- 1.19604
0.94	- 0.58122	1.28706	0.58271	- 1.29078
0.96	- 0.39406	1.35629	0.39504	- 1.35983
0.98	- 0.19902	1.39839	0.19951	- 1.40183
1.00	0.00000	1.41251	0.00000	- 1.41592

TABLE 3

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-SUPPORTED BEAM
Third Mode

$\frac{x}{\ell}$	ϕ_3	$\phi_3' = \frac{1}{\beta_3}\frac{d\phi_3}{dx}$	$\phi_3'' = \frac{1}{\beta_3^2}\frac{d^2\phi_3}{dx^2}$	$\phi_3''' = \frac{1}{\beta_3^3}\frac{d^3\phi_3}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00000
0.02	0.03886	0.36672	1.59173	- 1.99731
0.04	0.14410	0.65020	1.18532	- 1.97961
0.06	0.29879	0.85122	0.78508	- 1.93509
0.08	0.48626	0.97168	0.39742	- 1.85535
0.10	0.69037	1.01491	0.03009	- 1.73537
0.12	0.89584	0.98593	- 0.30845	- 1.57331
0.14	1.08857	0.89148	- 0.60968	- 1.37037
0.16	1.25604	0.74002	- 0.86560	- 1.13046
0.18	1.38759	0.54152	- 1.06927	- 0.85985
0.20	1.47476	0.30725	- 1.21523	- 0.56678
0.22	1.51147	0.04939	- 1.29988	- 0.26098
0.24	1.49419	- 0.21934	- 1.32168	0.04683
0.26	1.42202	- 0.48616	- 1.28137	0.34551
0.28	1.29662	- 0.73864	- 1.18195	0.62397
0.30	1.12212	- 0.96520	- 1.02863	0.87171
0.32	0.90489	- 1.15556	- 0.82867	1.07934
0.34	0.65324	- 1.30107	- 0.59110	1.23893
0.36	0.37703	- 1.39512	- 0.32637	1.34445
0.38	0.08727	- 1.43330	- 0.04596	1.39199
0.40	- 0.20439	- 1.41364	0.23807	1.37996
0.42	- 0.48616	- 1.33665	0.51362	1.30919
0.44	- 0.74658	- 1.20525	0.76897	1.18287
0.46	- 0.97504	- 1.02471	0.99330	1.00646
0.48	- 1.16223	- 0.80234	1.17711	0.78746
0.50	- 1.30050	- 0.54726	1.31263	0.53513
0.52	- 1.38422	- 0.26994	1.39411	0.26005
0.54	- 1.41001	0.01818	1.41807	- 0.02624
0.56	- 1.37687	0.30522	1.38344	- 0.31179
0.58	- 1.28624	0.57929	1.29160	- 0.58465
0.60	- 1.14194	0.82907	1.14631	- 0.83344
0.62	- 0.95000	1.04422	0.95356	- 1.04778
0.64	- 0.71844	1.21582	0.72134	- 1.21873
0.66	- 0.45691	1.33678	0.45927	- 1.33915
0.68	- 0.17628	1.40210	0.17821	- 1.40403
0.70	0.11174	1.40906	- 0.11017	- 1.41064
0.72	0.39519	1.35742	- 0.39391	- 1.35870
0.74	0.66227	1.24931	- 0.66123	- 1.25036
0.76	0.90188	1.08924	- 0.90103	- 1.09010
0.78	1.10404	0.88387	- 1.10335	- 0.88458
0.80	1.26035	0.64175	- 1.25980	- 0.64233
0.82	1.36432	0.37294	- 1.36386	- 0.37341
0.84	1.41160	0.08860	- 1.41124	- 0.08900
0.86	1.40025	- 0.19943	- 1.39996	0.19910
0.88	1.33072	- 0.47918	- 1.33049	0.47891
0.90	1.20590	- 0.73904	- 1.20573	0.73881
0.92	1.03098	- 0.96820	- 1.03085	0.96800
0.94	0.81323	- 1.15713	- 0.81313	1.15695
0.96	0.56168	- 1.29798	- 0.56162	1.29782
0.98	0.28680	- 1.38490	- 0.28677	1.38476
1.00	0.00000	- 1.41429	0.00000	1.41414

TABLE 3

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-SUPPORTED BEAM
Fourth Mode

$\frac{x}{l}$	ϕ_4	$\phi_4' = \frac{1}{\beta_4}\frac{d\phi_4}{dx}$	$\phi_4'' = \frac{1}{\beta_4^2}\frac{d^2\phi_4}{dx^2}$	$\phi_4''' = \frac{1}{\beta_4^3}\frac{d^3\phi_4}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00000
0.02	0.06496	0.46278	1.46633	- 1.99408
0.04	0.23451	0.78357	0.93792	- 1.95600
0.06	0.47104	0.96521	0.42662	- 1.86287
0.08	0.73820	1.01441	- 0.05091	- 1.70171
0.10	1.00204	0.94270	- 0.47581	- 1.46893
0.12	1.23237	0.76664	- 0.82947	- 1.16955
0.14	1.40407	0.50751	- 1.09559	- 0.81599
0.16	1.49825	0.19041	- 1.26206	- 0.42660
0.18	1.50306	- 0.15704	- 1.32223	- 0.02380
0.20	1.41422	- 0.50624	- 1.27577	0.36779
0.22	1.23502	- 0.82944	- 1.12901	0.72343
0.24	0.97582	- 1.10140	- 0.89466	1.02024
0.26	0.65324	- 1.30107	- 0.59110	1.23893
0.28	0.28879	- 1.41295	- 0.24121	1.36537
0.30	- 0.09274	- 1.42807	0.12917	1.39164
0.32	- 0.46510	- 1.34455	0.49299	1.31666
0.34	- 0.80250	- 1.16772	0.82386	1.14636
0.36	- 1.08150	- 0.90963	1.09785	0.89328
0.38	- 1.28266	- 0.58823	1.29518	0.57571
0.40	- 1.39201	- 0.22602	1.40160	0.21644
0.42	- 1.40200	0.15152	1.40934	- 0.15886
0.44	- 1.31209	0.51780	1.31771	- 0.52342
0.46	- 1.12877	0.84697	1.13308	- 0.85127
0.48	- 0.86513	1.11580	0.86843	- 1.11910
0.50	- 0.53994	1.30530	0.54246	- 1.30782
0.52	- 0.17628	1.40210	0.17821	- 1.40403
0.54	0.20000	1.39937	- 0.19853	- 1.40084
0.56	0.56222	1.29734	- 0.56109	- 1.29847
0.58	0.88466	1.10326	- 0.88379	- 1.10413
0.60	1.14445	0.83092	- 1.14379	- 0.83159
0.62	1.32317	0.49963	- 1.32266	- 0.50014
0.64	1.40813	0.13289	- 1.40774	- 0.13328
0.66	1.39330	- 0.24329	- 1.39301	0.24299
0.68	1.27973	- 0.60226	- 1.27950	0.60203
0.70	1.07546	- 0.91854	- 1.07529	0.91837
0.72	0.79497	- 1.16974	- 0.79484	1.16960
0.74	0.45814	- 1.33802	- 0.45804	1.33792
0.76	0.08884	- 1.41146	- 0.08876	1.41138
0.78	- 0.28676	- 1.38486	0.28682	1.38480
0.80	- 0.64202	- 1.26010	0.64206	1.26005
0.82	- 0.95176	- 1.04602	0.95180	1.04598
0.84	- 1.19405	- 0.75779	1.19407	0.75776
0.86	- 1.35168	- 0.41585	1.35170	0.41583
0.88	- 1.41351	- 0.04443	1.41352	0.04441
0.90	- 1.37513	0.33014	1.37514	- 0.33015
0.92	- 1.23928	0.68130	1.23929	- 0.68131
0.94	- 1.01558	0.98416	1.01559	- 0.98418
0.96	- 0.71989	1.21727	0.71990	- 1.21728
0.98	- 0.37317	1.36409	0.37318	- 1.36409
1.00	0.00000	1.41421	0.00000	- 1.41422

TABLE 3

CHARACTERISTIC FUNCTIONS AND DERIVATIVES
CLAMPED-SUPPORTED BEAM
Fifth Mode

$\frac{x}{l}$	ϕ_5	$\phi_5' = \frac{1}{\beta_5}\frac{d\phi_5}{dx}$	$\phi_5'' = \frac{1}{\beta_5^2}\frac{d^2\phi_5}{dx^2}$	$\phi_5''' = \frac{1}{\beta_5^3}\frac{d^3\phi_5}{dx^3}$
0.00	0.00000	0.00000	2.00000	- 2.00000
0.02	0.09685	0.55098	1.34119	- 1.98902
0.04	0.33974	0.88607	0.69424	- 1.92005
0.06	0.65851	1.01311	0.08494	- 1.75656
0.08	0.98717	0.95000	- 0.45262	- 1.48455
0.10	1.26755	0.72628	- 0.88320	- 1.11064
0.12	1.45308	0.38243	- 1.17672	- 0.65879
0.14	1.51200	- 0.03274	- 1.31329	- 0.16597
0.16	1.42950	- 0.46599	- 1.28662	0.32312
0.18	1.20840	- 0.86454	- 1.10567	0.76182
0.20	0.86819	- 1.18105	- 0.79432	1.10719
0.22	0.44239	- 1.37825	- 0.38928	1.32514
0.24	- 0.02533	- 1.43261	0.06352	1.39442
0.26	- 0.48616	- 1.33665	0.51362	1.30919
0.28	- 0.89158	- 1.09954	0.91132	1.07980
0.30	- 1.19872	- 0.74602	1.21291	0.73183
0.32	- 1.37505	- 0.31360	1.38526	0.30340
0.34	- 1.40200	0.15152	1.40934	- 0.15886
0.36	- 1.27698	0.59950	1.28226	- 0.60478
0.38	- 1.01369	0.98227	1.01748	- 0.98607
0.40	- 0.64067	1.25871	0.64340	- 1.26144
0.42	- 0.09828	1.39912	0.20024	- 1.40109
0.44	0.26570	1.38846	- 0.26429	- 1.38987
0.46	0.70119	1.22792	- 0.70018	- 1.22894
0.48	1.06118	0.93487	- 1.06045	- 0.93560
0.50	1.30682	0.54093	- 1.30630	- 0.54146
0.52	1.41161	0.08861	- 1.41124	- 0.08899
0.54	1.36423	- 0.37331	- 1.36395	0.37304
0.56	1.16977	- 0.79500	- 1.16957	0.79481
0.58	0.84919	- 1.13100	- 0.84905	1.13086
0.60	0.43706	- 1.34505	- 0.43696	1.34495
0.62	- 0.02218	- 1.41408	0.02225	1.41400
0.64	- 0.47902	- 1.33063	0.47907	1.33058
0.66	- 0.88421	- 1.10371	0.88425	1.10368
0.68	- 1.19405	- 0.75779	1.19407	0.75776
0.70	- 1.37513	- 0.33015	1.37515	0.33013
0.72	- 1.40793	0.13308	1.40794	- 0.13310
0.74	- 1.28892	0.58196	1.28892	- 0.58197
0.76	- 1.03091	0.96809	1.03092	- 0.96810
0.78	- 0.66175	1.24983	0.66176	- 1.24984
0.80	- 0.22123	1.39680	0.22123	- 1.39680
0.82	0.24314	1.39315	- 0.24314	- 1.39316
0.84	0.68130	1.23928	- 0.68130	- 1.23929
0.86	1.04600	0.95178	- 1.04600	- 0.95178
0.88	1.29790	0.56165	- 1.29790	- 0.56165
0.90	1.40985	0.11096	- 1.40985	- 0.11096
0.92	1.36978	- 0.35170	- 1.36978	0.35170
0.94	1.18201	- 0.77644	- 1.18201	0.77644
0.96	0.86678	- 1.11745	- 0.86678	1.11745
0.98	0.45809	- 1.33797	- 0.45809	1.33797
1.00	0.00000	- 1.41421	0.00000	1.41421

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